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Generalization of Classical Nonlocal Boundary Values Problems for Elliptic Equations

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Abstract

In this paper, the solution of classical and generalized linear and nonlinear elliptic partial differential equations of the second order with non-local boundary conditions is studied. The basic concepts and theorems related to the study are included, which includes the main result of this research. Many basic definitions, theorems and observations about Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ are discussed. The existence and uniqueness of the solution of the Poisson equation with non-local boundary conditions are considered. The existence and uniqueness of the solution of a second-order quasi-linear elliptic differential equation with non-linear integral boundary condition are also discussed. The argument for proving the previous problem is based on Banach's fixed point theorem in the complete metric space, the maxima and minima principle, and the comparison principle.

Key word: Sobolev Spaces, Poisson Equation, Quesilinear Ellipitic Differential Equation, Boundary, and Banach's fixed Point Theorem.

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تعميمات مسائل القيم الحدية غير الموضعية للمعادلات الاهليجية

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الملــخــص

في هذه الورقة، تمت دراسة حل المعادلات التفاضلية الجزئية الإهليلجية الخطية وغير الخطية الكلاسيكية والمعممة من الدرجة الثانية مع شروط حدودية غير محلية. كما تم تضمين المفاهيم الأساسية والنظريات المتعلقة بالدراسة، والتي تشمل النتيجة الرئيسية لهذا البحث. وأيضا نوقشت العديد من التعريفات الأساسية والنظريات والملاحظات حول فضاءات سوبوليف $H_0^k(\Omega)$, $H^k(\Omega)$. كما اشتملت هذه الورقة على دراسة وجود حل متفرد لمعادلة بواسون مع شروط حدودية غير محلية. وتمت أيضًا مناقشة وجود حل متفرد للمعادلة التفاضلية الإهليلجية شبه الخطية من الدرجة الثانية مع شرط حدودي متكامل غير خطي. بحيث تستند الحجة لإثبات المشكلة السابقة إلى نظرية النقطة الثابتة لباناخ في الفضاء المتري الكامل، ومبدأ الحد الأقصى والحد الأدنى، ومبدأ المقارنة. كما ان أسلوب البرهان في المسائل يعتمد اساسا علي كل من نظرية النقطة الثابتة لباناخ في الفضاء المتري الكامل– مبدأ الحد الأقصى والحد الأدنى– مبدأ المقارنة.

الكلمات المفتاحية: فضاء سوبوليف، معادلة بواسن، المعادلة التفاضلية الاهليجية الخطية، و معادلة النقطة الثابتة لبانخ.

Introduction

This work presents results concerning second-order elliptic partial differential equations defined over a domain $\boldsymbol{\Omega} \subset \mathbb{R}_{\boldsymbol{n}}$, with nonlocal boundary conditions in the context of a Dirichlet problem. Unlike the classical boundary value problem, where boundary conditions establish relationships between the values of the unknown function or its derivatives at the same boundary point, the nonlocal boundary value problem introduces conditions linking the values of the function or its derivatives at different points within Ω^{-} .This distinctive feature broadens the applicability of the problem and poses unique analytical challenges. The study explores the mathematical formulation, analysis, and potential solutions under these nonlocal conditions, contributing to the theoretical understanding and practical applications of such problems in mathematical physics and engineering.

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In [1] Carleman T. addressed the problem of finding a holomorphic function that satisfies nonlocal boundary conditions. These conditions establish a connection between the values of the unknown function at a point $t \in \partial\Omega$ and another point $\alpha(t) \in \partial\Omega$ where $\alpha\big(\alpha(t)\big) = t$ and $\alpha(\partial\Omega) = \partial\Omega$.
In [2], Bitsadze A.V. and Samarskij A.A. studied the Laplace equation with nonlocal boundary conditions. These conditions prescribe a connection between the trace of an
unknown function on a manifold $\Gamma_1 \subset \partial\Omega$ and its trace on another manifold $\Gamma_0 \subset \Omega$ while imposing a first boundary condition on $\partial\Omega\backslash\Gamma_1$.
In [3], Chabrowski investigated a class of nonlocal problems involving linear elliptic boundary conditions. Skubachevskij, in [4], examined general linear elliptic equations of order 2nd under generalized nonlocal boundary conditions. Nonlocal boundary conditions for ordinary differential equations have also been studied extensively, highlighting their importance in various applications. Nonlinear elliptic equations with nonlocal and nonlinear boundary conditions were explored by L. Simon in [5] and [6], and by I.M. Hassan in [7], [8], and [9]. Classical nonlocal and nonlinear first boundary value problems for elliptic partial differential equations were analyzed by I.M. Hassan in [7].
The existence and uniqueness of the solution of the second-order partial differential equations for foam are studied:

$$\Delta u = f \quad in \quad \Omega \tag{1}$$

subject to the following new nonlocal boundary conditions:

$$u(x) = h(x)u\big(\Phi(x)\big) + \Psi(x)\,; \quad x \in \partial\Omega \tag{2}$$

where $\Omega \subset \mathbb{R}_n$ is a bounded domain, h is a continuous function on $\partial\Omega$ satisfying $sup \mid h \mid < 1$, and $\Phi: \partial\Omega \to \Omega^-$ is a continuous mapping. We also address the existence and uniqueness of solutions for the problem:

$$Q(u) = \sum_{i,j=1}^{n} a_{ij}(x,u,\partial u)\partial_i\partial_j u + b(x,u,\partial u) = 0 \text{ in } \Omega \tag{3}$$

subject to the boundary condition:

$$u(x) := h\,(x, u\,\big(\Phi(x)\big), x \in \partial\Omega \tag{4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\Phi : \partial\Omega \to \overline{\Omega}$ is a continuous mapping and $h: \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is continuous with $\partial_2 h$ satisfying sup $sup|\partial_2 h| < 1$. Finally, we establish the existence and uniqueness of solutions for the equation:

$$Q(u) = \sum_{i,j=1}^{n} a_{ij}(x,u,\partial u)\partial_i\partial_j u + b(x,u,\partial u) = 0 \text{ in } \Omega \tag{5}$$

with the nonlocal boundary condition:

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$$u(x) = h_1\left(x, u(\Phi(x)) + \int_{\partial\Omega} h_2(x,t,u(\Psi(t))\right) d\sigma \text{ on } \partial\Omega \qquad (6)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain $x = (x_1, x_2, \ldots\ldots, x_n) \in \Omega$, $n \geq 2$ and $u \in C^2(\Omega)$. The coefficients $a_{ij}(x,z,p), (i,j = 1, \ldots., n), b(x,z,p)$ are assumed to be real valued and defined for all values of (x,z,p) in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, further $a_{ij} = a_{ji}$ .we expect that the reader is familiar with topics in real analysis. we provide two counterexamples illustrating that if the conditions $sup|h| < 1$ and $sup|\partial_2 h| < 1$ are not satisfied, the nonlocal and nonlinear first boundary value problem may lack a solution or admit multiple solutions.

1.1 Background

We recall the following definitions:

Definition 1.1.1 [4] (Banach's fixed point) A fixed point of mapping $T: X \to X$ of a set X in to itself an $x \in X$ which is mapped in to itself(is " kept fixed" by T) that is, $T_x = x$, The image coincides with x.

Theorem 1.1.2 [4] (Banach's fixed point) Consider a metric space $X = (X, d)$, where X $\neq \emptyset$. Suppose that is X complete and let $T: X \to X$ be contraction on X. Then T has precisely one fixed point.

Theorem 1.1.3 [9] (Maximum Principle) Assume that $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C(\bar{\Omega})$ is harmonic in Ω , If there exists a point $x_0 \in \Omega$ (inside) such that Ω is compact set , where $u(x_0) = max_{\bar{\Omega}}\, u = sup_{\bar{\Omega}}\, u$ then u is constant . Except of this case u it has its maximum only on the boundary.

Remark: Similarly the minimum principle can be formulated; that is $If \,\exists\, x_0 \in \Omega : u(x_0) = min_{\bar{\Omega}}\, u$ then u is constant.

The Comparison Principle [9]: If Q is (a linear operator) satisfying the hypotheses of the weak maximum principle (see [11]in page 32) and if $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy inequalities $Qu \geq Qv$ in Ω, and $u \leq v$ on $\partial\Omega$ we have immediately from corollary 3.2 in page 33 in [11] that $u \leq v$,this Comparison princ-iple has the following extension to quasilinear operators .

Theorem 1.1.4 [9] Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying an exterior sphere condition at each point of the boundary $\partial\Omega$, let Q be a divergence structure operator with coefficients $A^i \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n), i = 1, \ldots\ldots, n, B \in C^y(\Omega \times \mathbb{R} \times \mathbb{R}^n)$;
$0 < y < 1$, Satisfying the hypotheses of Theorem 15.8, together with the hypotheses of theorem 10.9 for $\alpha = \tau + 2$. Then, for any

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function $\varphi \in C^0(\partial\Omega)$, there exists a solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of the Dirichlet problem $Qu = 0$ in $\Omega, u = \varphi$ on $\partial\Omega$.

Theorem 1.1.5 (Lagrange's Mean value) Let f be continues function, such that $\frac{\partial f}{\partial y}$ exists and bounded in Ω. Then for all $(x, y_1), (x, y_2) \in \Omega$ there exists $l \in R$ such that the following equality holds :

$$f(x, y_1) - f(x, y_2) = \frac{\partial f}{\partial y}(x, y_1 + l(y_2 - y_1).(y_1 - y_2)$$

1.2 Some Results on Partial Differential Equations

Definition 1.2.1 [10] Let $\Omega \subset \mathbb{R}^n$ be a domain$(n \leq 2)$, and $F(x, \dots, p_{\propto 1, \dots\dots \propto_n}, \dots.)$ be a given real function the points x belonging to the domain Ω and some real variables $p_{\propto 1, \dots \propto_n}$ with nonnegative indices $\propto_1, \dots \dots \propto_n$ $(\sum_{i=1}^{n} \propto_i = \gamma, \gamma = 0, \dots, m; m \geq 1)$, suppose that $\sum_{i=1}^{n} \propto_i = m$ at least one of the derivatives $\frac{\partial F}{\partial p_{\propto_1, \dots\dots, \propto_n}}$ of the function F is different from zero. An equality of the from:

$$F\left(x, \dots., \frac{\partial^\gamma u}{\partial x_1{}^{\propto_1} \dots \partial x_n^{\propto_n}}\right) = 0 \qquad (7)$$

is called a partial differential equation of order n , with respect to the unknown function $u(x) = u(x_1, \dots., x_n), x \in \Omega$; the left hand member of this equality is called a partial differential operator of the m-th order . A real function $u\,(x)$ defined in the domain Ω , where equation (7) considered, which turns the equation into an identity, is called a regular solution of the equation.

Definition 1.2.2 [10] (linear Equations): Equation (1) is said to be linear if F is linear function with respect to all variable $p_{\propto_1, \dots\dots, \propto_n}(\sum_{i=1}^{n} \propto_n = \gamma, \gamma = 0, \dots, m; m \geq 1)$.

Definition 1.2.3 [10] (quasi-linear Equations): Equation (7) is said to be quasi-linear when F is linear with respect to the variable $p_{\propto_1, \dots\dots, \propto_n}$ *only for* $\sum_{i=0}^{n} \propto_i = m$.

1.3 Classification of Partial Differential Equation

Some linear second order partial differential equations can be classified as parabolic, hyperbolic or elliptic.

1.3.1 Equation of second order [10] Assuming $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, the general second order partial differential equation in two independent variables has the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x \partial} + E\frac{\partial u}{\partial y} + Fu = G$$

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where the coefficients A, B, C, D, E, F, G may depend on x and y. This form is analogous to the equation for a conic section.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = G$$

Just as one classifies conic sections into parabolic, hyperbolic and elliptic based on the discriminate, $B^2 - 4AC$, the same can be done for a second-order partial differential equation at a given point.

i- $B^2 - 4AC < 0$ (Elliptic partial differential equation).

ii- $B^2 - 4AC = 0$ (parabolic partial differential equation).

iii- $B^2 - 4AC > 0$ (hyperbolic partial differential equation).

If there are n independent variables $x_1, \dots, x_n$ a general linear partial differential equation of a second-order has the form

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_j \frac{\partial u}{\partial x_j} + Cu = F \qquad (8)$$

where $A_{i,j}, B_j, C$ *and* F are real function of the variable point $x \in \Omega$.

2.1 $\mathbf{H^k(\Omega)}$ & $\mathbf{H_0^k(\Omega)}$ Spaces.

In this chapter we shall introduce sobolve spaces of integer order and establish some of their properties . We give some definitions and theorems without proof ,the proof can be found in [12]and [13].

Definition 2.1.1 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $K \geq 0$ be an integer. Consider the vector space of function, $f \in C^K(\overline{\Omega})$ define in $C^K(\Omega)$, the scalar product given by the formula: $< f, g > = \sum_{|\alpha| \leq k} \int_\Omega (\partial^\alpha f)(\partial^\alpha \overline{g})$, $f, g \in C^K(\bar{\Omega})$ Then we obtain the Euclidean space. The completion of this Euclidean space is called $H^K(\Omega)$ (Sobolev space).

What does mean completion?

If we denote this Euclidean space by X, The completion of X is a Hilbert space $\widetilde{X}$, such that it has a dense subset $\widetilde{x_0}$ which is isomorphic to X. $\widetilde{X} = H^K(\Omega)$, Thus $H^K(\Omega)$ is a Hilbert space.

Theorem 2.1.2 The space $H^K(\Omega)$ is isomorphic to the space X_1 of functions $f \in L^2(\Omega)$ such that $\partial f \in H^k(\Omega)$ in distributional sense if $|\alpha| \leq K$ there exist $f_j \in C^K\overline{(\Omega)}$ such that: $\lim_{j \to \infty} \|f - f_j\|_{x_1} = 0$, where inner product and $\|.\|$ in X_1 is defined by:

$$\langle f, g \rangle = \sum_{|\alpha| \leq k} \int_\Omega (\partial^\alpha f)(\partial^\alpha \overline{g}), \text{ and; } \|f\| = \left\{\sum_{|\alpha| \leq k} \int_\Omega |\partial f|^2\right\}^{½}$$

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Definition 2.1.3 let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $K \geq 0$ be an integer, Consider the vector space of functions, (f_j) , $f \in C_0^k(\overline{\Omega})$ define in $C^K(\Omega)$, the inner product given by the formula: $< f, g > = \sum_{|\alpha| \leq k} \int_\Omega (\partial^\alpha f)(\partial^\alpha \overline{g})$, Then we obtain an Euclidean space. The completion of this space is called) Soblove space, and donated $H_0^k(\Omega)$.

Theorem 2.1.4 The $H_0^k(\Omega)$ is isomorphic to X_2, where X_2 is space of function $f \in L^2(\Omega)$, $\partial f \in L^2(\Omega)$ if $|\alpha| \leq k$ and there exist (f_j) , $f_j \in C_0^k(\Omega)$ such that; $\lim_{j\to\infty} \|f\_f_j\|_{x_2} = 0$, where the scalar product and $\|.\|$ is defined by:

$$< f, g > = \sum_{|\alpha| \leq k} \int_\Omega (\partial^\alpha f)(\partial^\alpha \overline{g}), \text{ and } \|f\| := \sqrt{\langle f, f \rangle}$$

Remarks 2.1.5 In the case $k= 0$, $H^0(\Omega) = H_0^0(\Omega) = L^2(\Omega)$ Later we shall see that for $K > 0$, $H_0^k(\Omega)$ is a real subspace of $H^K(\Omega)$, $(H_0^k(\Omega) \neq H^K(\Omega))$.

2.2 Some Properties of $H^K(\Omega)$and $H_0^k(\Omega)$.

1- For all $\varphi \in C^K(\overline{\Omega}), f \in H^K(\Omega)$ implice that $\varphi f \in H^K(\Omega)$ and $\|\varphi f\|_{H^k(\Omega)} \leq C(\varphi)\|f\|_{H^k(\Omega)}$, φ is fixed, further if $f \in H_0^k(\Omega)$ implice that $\varphi f \in H_0^k(\Omega)$, and $\|\varphi f\|_{H_0^k(\Omega)} \leq C(\varphi)\|f\|_{H_0^k(\Omega)}$.

2- Let $0 \leq k < L$, $H^L(\Omega) \subset H^K(\Omega)$ and $H_0^L(\Omega) \subset H_0^k(\Omega)$, For all $f \in H^L(\Omega)$;
$\|f\|_{H^L(\Omega)} \geq \|f\|_{H^k(\Omega)}$.

3- Assume $\Omega \subset \Omega_1$ is a bounded domain of $\mathbb{R}^n$, Then for all $k \geq 0$ (integer), If $f \in H^K(\Omega_1) \Rightarrow f_{|\Omega} \in H^K(\Omega)$ and, $\|f\|_{H^k(\Omega_1)} \geq \|f\|_{H^k(\Omega)}$, Because $f \in H(\Omega_1)$ means that $f \in L^2(\Omega_1), \partial f \in L^2(\Omega_1)$ (for all α , $|\alpha| \leq k$), and there exist $f_j \in C^K(\Omega_1)$ such that: $\lim_{j\to\infty} \|f_j - f\|_{H^k(\Omega^1)} = 0$ implies that $f_{|\Omega} \in L^2(\Omega)$; $\partial^\alpha(f_{|\Omega}) \in L^2(\Omega), f \in C^K(\overline{\Omega})$, $\lim_{j\to\infty} \|f_j - f\|_{H^k(\Omega)} = 0$ $f \in H_0^k(\Omega_1) \nrightarrow f_{|_j} \in H_0^k(\Omega)$, Because $g \in C_0^k(\Omega) \Rightarrow g_{|\Omega} \in C_0^k(\Omega)$.

4-Let $\Omega \subset \Omega_1 \subset \mathbb{R}^n$ (bounded domain) and assume that $f \in H_0^k(\Omega)$ The extending f to Ω_1 as 0 out of Ω ($f(x) = 0$ if $x \in (\Omega_1 \backslash \Omega)$, we obtain a function $f \in H_0^k(\Omega)$ and, $\|f\|_{H_0^k(\Omega)} \geq$

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$\|f\|_{H_0^k(\Omega)}$, Because for $g \in C_0^k(\Omega)$ to Ω_1 if we extendg to Ω_1 as 0 out of Ω then; $g \in C_0^k(\Omega_1)$.

5-Assume that $f \in \widetilde{H}^K(\overline{\Omega})$, i.e. $f \in L^2(\Omega)$ and $\partial f \in L^2(\Omega)$ if $|\alpha| \leq k$, Define for $\forall\, \varepsilon > 0$, f_ε in the following way; $f_{\varepsilon(x)} =: \int_\Omega f(y)\,\eta_\varepsilon(x-y)dy, x \in \Omega, where\ \eta_\varepsilon \in C_0^\infty(R^n)$; Supp $\eta_\varepsilon \subset \bar{B}_\varepsilon, \eta_\varepsilon > 0, \int \eta_\varepsilon = 1$, Then for any $\Omega_0, \overline{\Omega_0} \subset \Omega$, we have $\lim\limits_{\varepsilon\to\infty}\|f_\varepsilon - f\|_{\widetilde{H}^k(\Omega)} = 0$.

6- As consequence of (5) we obtain that if $u \in \widetilde{H}^K(\Omega)$ and $u(x) = 0$ a. e out of compact subset of Ω , then; $u \in H_0^k(\Omega), u\,\varepsilon \in C_0^\infty(\Omega) \subset C_0^k(\Omega)$, for sufficiently small ε, $\lim\limits_{\varepsilon\to\infty}\|u_\varepsilon - u\|_{\widetilde{H}^k(\Omega)} = 0$.

7- Assume that $\Omega \in \mathbb{R}^n$ is a star like domain (bounded), i.e. there exist $x_0 \in \Omega$: $\forall\, 0 < \lambda < 1$

$$\overline{\left\{x \in \Omega : x_0 + \frac{x - x_0}{\lambda} \quad \in \Omega\right\}} \subset \Omega$$

$\partial\Omega$ in sufficiently smooth (piecewise continuous differentiable), $u \in C^1(\Omega), u|_{\partial\Omega} = 0$ This is Implies that $u \in H_0^1(\Omega)$.

2.3 Equivalent Norms in $H_0^1(\Omega)$.

We know that in $H_0^1(\Omega)$ we can define two norms as following by:
$\|f\| := \{\int_\Omega (|f|^2 + \sum_{j=1}^n |\partial_j f|^2)\}^{½}$, $\|f\|' := \{\int_\Omega \sum_{j=1}^n |\partial_j f|^2\}^{½}$.

Theorem 2.3.1 If $\Omega \subset \mathbb{R}^n$ is a bounded domain, then $\|.\|$ and $\|.\|'$ are equivalent norms in $H_0^1(\Omega)$, that is exist $c > 0$ such that: $c\|f\| \leq \|f\|' \leq \|f\|$ for all $f \in H_0^1(\Omega)$

Proof : **a)**- $\|f\|' \leq \|f\|$ is trivil to prove that $c\|f\| \leq \|f\|'$, There exist $c_1 > 0$ such that : **(1)** - $\int_\Omega |f|^2 \leq c_1 \int_\Omega \sum_{j=1}^n |\partial_j f|^2$ for all $f \in H_0^1(\Omega)$, This is implies: $\|f\|^2 = \int_\Omega |f| + \int_\Omega \sum_{j=1}^n |\partial j f|^2 \leq (c^1 + 1)\int_\Omega \sum_{j=1}^n |\partial j f|^2 = (c^1 + 1)\|f\|'^2$.

b)- Since Ω is bounded , there exist an interval $T = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$ In $\mathbb{R}^n$ such that $\Omega \subset$ T. By using property (a), extending function $f \in H_0^1(\Omega)$ to T as 0 out of Ω, we obtain function $f \in H_0^1(T)$, $\|f\|_{H_0^1(T)} = \|f\|_{H_0^1(\Omega)}$; $\|f\|`_{H_0^1(T)} = \|f\|`_{H_0^1(\Omega)}$ Therefor, instead of (a) it is sufficient to prove

(2)- $\int_T |f|^2 \leq c^1 \int_T \sum_{j=0}^n |\partial_j f|^2$, $\quad for\ all\ f \in H_0^1(T)$

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c)- Firstly we prove (b) for $f \in C_0^1(T)$ Noted, $x = (x_1, \dots, x_n), x = (x_1, x`)$, $x = (x_2, \dots, x_n)$, By the notation Laibainz form we have:

$f(x_1, x`) - f(a_1, x) = \int_{a_1}^{x_1} \partial^1 f(t, x`)dt$, $f \in C_0^1(T), f(x_1, x`) = \int_{a_1}^{x_1} \partial_1 f(t, x`)dt$

Implies that:

$$|f(x_1, x`)|^2 = \left|\int_{a^1}^{x_1} 1.\partial^1 f(t, x`)dt\right|^2 \le \left(\int_{a_1}^{x_1} 1^2 dt\right).\int_{a_1}^{x_1} |\partial_1 f(t, x`)|^2 dt = (x_1 - a_1)\int_{a_1}^{x_1} |\partial_1 f(t, x`)|^2 dt \le (x_1 - a_1)\int_{a_1}^{b_1} |\partial_1 f(t, x`)|^2 dt.$$

$$\left|\int_T f\right|^2 = \int_{a_1}^{b_1} \left[\left|\int_T f(x_1, x`)\right|^2 dx`\right] dx_1$$
$$\le \int_{a_1}^{b_1} \left\{\int_{T`} \left[(x_1 - a_1)\int_{a_1}^{b_1} |\partial_1 f(t, x`)|^2 dt\right] dx`\right\} dx_1$$
$$\le \frac{(v_1 - a_1)^2}{2} \int_T |\partial^1 f|^2 \le c^1 \int_T \sum_{j=0}^n \left|\partial_j f\right|^2;$$

where $T` = (a_2, b_2) \times \dots \times (a_n, b_n)$.

d)-We prove (2) for $f \in H_0^1(T)$ we know that $\to f_j \in C_0^1 T)$ $\lim_{j\to\infty} \|fj - f\|_{H_0^1(T)} = 0$, By c, (2) in vcolid for f_j.

$\int_T |fj|^2 \le c_1 \int_T \sum_{i=0}^n \left|\partial_j f_j\right|^2, J \to \infty$,Because $f_j \to f$ in $L^2(T)$; $\int_T |f|^2 \int_T \sum_{j=0}^n \left|\partial_j f_j\right|^2$; Because $\partial_j f_j \to \partial_j t$,∴ $L^2(T)$ Implies that $\int_T |f|^2 \le c_1 \int_T \sum_{j=0}^n \left|\partial_j f\right|^2$.

Remark 2.3.2 In $H^1(\Omega)$ the above type of theorem is not true, because for $f, f(x) = 1, \forall x \in \Omega$, we have; $\|f\|^2 = \int_\Omega |f|^2 + \sum_{j=0}^n \left|\partial_j f\right|^2 = \int_\Omega 1 > 0$, But $\|f\|^2 = \int_\Omega \sum_{j=0}^n \left|\partial_j f\right|^2 = 0$, This example shows also that $H_0^1(\Omega)$, is a real subspace of $H^1(\Omega)$ because $f \in H^1(\Omega), f \notin H_0^1(\Omega)$.

Definition 2.3.3 [14] (Completely Continuous) The operator $A: H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is completely continuous if for all bounded sequence $(f_j) \in H_0^1(\Omega)$, there exists sub a sequence $(f_j)`$ which converges to $L^2(\Omega)$.

2.4 Imbedding of $H_0^1(\Omega)$)and $H^1(\Omega)$ into $L^2(\Omega)$.

Definition 2.4.1 The imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is an Operator $A: H_0^1(\Omega) \to L^2(\Omega)$ such that; for all $f \in H_0^1(\Omega)$, $Af = f \in L^2(\Omega)$.

Theorem 2.4.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain then the imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is a completely continuous

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operator, further if $\partial\Omega$ is a piecewise continuous differentiable, then the imbedding of $H^1(\Omega)$ into $L^2(\Omega)$ is also a completely continuous operator.

Definition 2.4.3 Trace of function of $\mathbf{H^1(\Omega)}$ The operator $\widetilde{L}: H^1(\Omega) \longrightarrow L^2(\partial\Omega)$ is called the trace operator and for any $f \in H^1(\Omega)$, $\tilde{L}f \in L^2(\partial\Omega)$ is called the trace of f on $\partial\Omega$.

Theorem 2.4.4 Assume that $\Omega \subset \mathbb{R}^n$ is abounded domain, $\partial\Omega$ is piecewise continuous differentiable, then there exist $C > 0$ such that: $\|f\|^2_{L^2(\Omega)} = \int_{\partial\Omega} |f|^2 d\sigma \le C\|f\|^2_{H^1(\Omega)}$ $\forall f \in C^1(\bar{\Omega})$ Consider the following operator L such that: $Lf = f_{|\partial\Omega}$, $f \in C^1(\bar{\Omega})$, Cleary, L a linear operator, further in virtue to above theorem we know that L is bonded operator $H^1(\Omega)$, in to $L^2(\Omega)$, L is not defined in whole $H^1(\Omega)$ but in $C^1(\bar{\Omega})$ which is dense in $H^1(\Omega)(\overline{C^1(\bar{\Omega})} = H^1(\Omega)$ then L can be uniquely extended to $H^1(\Omega)$ (the extension of L) will be denoted by $\tilde{L}$ such that; $\widetilde{L}: H^1(\Omega) \longrightarrow L^2(\partial\Omega)$, is a linear and bounded operator.

Definition 2.4.5 The operator $\widetilde{L}: H^1(\Omega) \longrightarrow L^2(\partial\Omega)$ is called the trace operator and for any $f \in H^1(\Omega)$, $\tilde{L}f \in L^2(\partial\Omega)$ is called the trace of f on $\partial\Omega$.

Remark 2.4.6

1)-By the definition, if $f \in C^1(\bar{\Omega})$ then the trace of $f(i.e. \tilde{L} f)$ is equal to the restriction of f on $\partial\Omega$ i.e $f_{|\partial\Omega}$; $\tilde{L}f = f|_{\partial\Omega}$ Further sequel we shall denote the trace of f (for arbitrary $f \in H^1(\Omega)$) by $f|_{\partial\Omega}$

2)-The trace of $H^1(\Omega)$ can be defined directly in the following manner: Consider The sequence (f_j) of function $f_j \in C^1(\bar{\Omega})$ such that, $\lim_{j\to\infty} \|f_j - f\|_{H^1(\Omega)} = 0$ Consider the restriction of f_j on $\partial\Omega(f_{j_{|\partial\Omega}})$. The sequence $(f_{j_{|\partial\Omega}})$ is a Cauchy sequence in $L^2(\Omega)$, ($L^2(\Omega)$ is complete); Its limit is the trace of f on $\partial\Omega$.

3)- If $f \in H^1_0(\Omega)$ then for $f_{|\partial\Omega}$ (the trace), $f_{|\partial\Omega} = 0$. Fueled, using **(2)**,there exist $(f_j) f_j \in C^1_0(\Omega). \lim_{j\to\infty} \|f_j - f\|_{H^1_0(\Omega)} = 0$, But $(f_{j_{|\partial\Omega}}) = 0, \to \lim_{j\to\infty} (f_{j_{|\partial\Omega}}) = 0$, in $L^2(\Omega)$ i. e $f_{|\partial\Omega} = 0$, It can be proved that the in true if $f \in H^1_0(\Omega)$ and $f_{|\partial\Omega} = 0$; implice that $f \in H^1(\Omega)$. Assume $\partial\Omega$ is a piecewise continuous differentiable, that is, for $f \in H^1(\Omega), f_{|\partial\Omega} = 0$ if and only if $f \in H^1_0(\Omega)$.

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3.1 Nonlocal Elliptic First Boundary Value Problem.
The aim of this part is to prove existence of solution of second order partial differential equations in a domain $\Omega \subset \mathbb{R}^n$ with the following nonlocal boundary conditions:

$$u(x) = h(x)u\big(\Phi(x)\big) + \Psi(x);\;\; x \in \partial\Omega \quad (1),$$

$$u(x) = h\left(x, u\big(\Phi(x)\big)\right),\;\; x \in \partial\Omega \quad (2)$$

$$u(x) = h_1\big(x, u(x)\big) + \int_{\partial\Omega} h_2\left(x, t, u\big(\Psi(x)\big)\right) d\sigma_t;\; x \in \partial\Omega \; (3)$$

and

$$\partial_{\nu^*} u := h_1\big(x, u(x)\big) + h_2\left(x, u\big(\Phi(x)\big)\right) + \int_{\partial\Omega} h_3\left(x, t, u\big(\Psi(t)\big)\right) d\sigma_t \in \partial\Omega \quad (4).$$

where $\Omega \subset \mathbb{R}^n$is a bounded domain and $\Phi, \Psi$are given continuous mapping from $\partial\Omega$ in to $\overline{\Omega}$.

3.2 Classical Nonlocal First Boundary Value Problem.
We shall prove the existence and uniqueness of the solution of Poisson's equation with new non-local boundary condition (1), that is, we prove the existence and uniqueness of the solution of second order partial differential equation:

$$\Delta u = f \;\; in \;\; \Omega, \quad (5)$$

a)- h: is a given continuous function on the boundary $\partial\Omega$ such that $sup|h| < 1$ and Φ: $\partial\Omega \to \overline{\Omega}$ is a continuous mapping.
Definition 3.2.1 The classical nonlocal first boundary value problem for the poisons equation is the following:

$$\begin{cases} \Delta u = f \; in \;\; \Omega \; (\Delta u = \sum_{i,j=1}^{n} \partial_i \partial_j u)\,, & (6) \\ u(x) = h(x)u\big(\Phi(x)\big) + \psi(x),\; x \in \partial\Omega, & (7) \end{cases}$$

where $U \in C^2(\Omega) \cap C(\overline{\Omega})$
Assume that f and $\partial\Omega$ are sufficiently smooth such that the solution of the usual Dirichlet problem:

$$\begin{cases} \Delta V = f \; in \, \Omega & (8) \\ V_{|\partial\Omega} = \varphi \;\; on \quad \partial\Omega & (9) \end{cases};$$

Exists for any arbitrary $\varphi \in C(\partial\Omega)$.
Theorem 3.2.2 Assume that the conditions (a) in equation (5) are fulfilled, and then the classical nonlocal boundary value problem (6), (7) has a unique solution.
Proof : Denote by $G(\varphi)$ the unique solution V of problem (8), (9) for ther define an operator A by:

$$A(\varphi)(x) = h(x)[G\,(\varphi)]\big(\Phi(x)\big) + \psi(x)\,; x \in \partial\Omega \qquad (10)$$

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Then $A: C(\partial\Omega) \to C(\partial\Omega)$ is a nonlinear mapping, $C(\partial\Omega)$ is a contraction on the complete metric space.

$$\rho(\varphi, \tilde{\varphi}) = sup_{\partial\Omega} |\varphi - \tilde{\varphi}| \qquad (11)$$

If $\varphi \in C(\partial\Omega)$ fixed point of A, i.e $A(\varphi) = \varphi$,Then $u = G(\varphi)$ is a solution of problem (6), (7). Conversely, if u is a solution of (2), then: $\varphi = u|_{\partial\Omega}$ is a fixed point of A. Therefore, to prove existence and uniqueness of the solution for problem (6), (7), it is sufficient to show that A has exactly one fixed point. This will be a consequence of Banach's fixed point theorem [15]. New, we prove that:$A: C(\partial\Omega) \to C(\partial\Omega)$ is a contraction on the complete metric space c$(\partial\Omega)$;

$$p\big(A(\varphi)(x) - A(\tilde{\varphi})(x)\big) = sup|A(\varphi)(x) - A(\tilde{\varphi})(x)| \leq p.q\,(\varphi, \tilde{\varphi}) \qquad (12)$$

where q is a non-negative number. Since $G(\varphi) - G(\tilde{\varphi})$ is a solution of the Laplace equation, then by maximum principle we find:

$$|A(\varphi)(x) - A(\tilde{\varphi})(x)| = |h(x)[G(\varphi)(\Phi(x) - G(\varphi)(\Phi(x))]| \leq sup_{\partial\Omega}|h|\|\varphi - \tilde{\varphi}\|_{c(\partial\Omega)} \qquad (13)$$

Consequently; $sup|A(\varphi)(x) - A(\tilde{\varphi})(x| \leq sup_{\partial\Omega} \|\varphi - \tilde{\varphi}\|_{c(\partial\Omega)}$

i.e. $|A(\varphi)(x) - A(\tilde{\varphi})(x)|_{c(\partial\Omega)} \leq q.\|\varphi - \tilde{\varphi}\|_{c(\partial\Omega)}$ (14) where $q = sup|h| \leq 1$. This is equivalent to equation (10): $\rho\big(A(\varphi), A(\tilde{\varphi})\big) \leq q.\rho(\varphi, \tilde{\varphi})$ which means that A is a contraction on the complete metric space $C(\partial\Omega)$, and this there exists a unique $\varphi \in C(\partial\Omega)$ such that $A(\varphi) = \varphi$, that is, $u = G(\varphi)$ will be the unique solution of (6), (7).

Remark 3.2.3 The condition (a) is not fulfilled then the non-local boundary value problem may have no solution or it may have several solutions.

3.3 Nonlocal first Boundary Value Problem For quasi-linear partial differential equations:

We shall consider the of the from:

$$Q(u) \coloneqq \sum_{i,j=1}^{n} a_{ij}(x, u, \partial u)\partial_i\partial_j u + b(x, u, \partial u) = 0 \ \ in\ \Omega \qquad (15)$$

$$u(x) = h\big(x, u(\Phi(x)\big), \quad x \in \partial\Omega \qquad (16)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

b) $\Phi: \partial\Omega \longrightarrow \bar{\Omega}$ is a continuous mapping $h: \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the property $sup|\partial_2 h| < 1$. We shall prove existent and uniqueness of the solution problem (15),(16). Its proved the following comparison principle (see Theorem 10.1 of [16]. Let Q be a second order quasilinear elliptic operator defined by the formula:

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$Q(u) \coloneqq \sum_{i,j=1}^{n} a_{ij}(x,u,\partial u)\partial_i\partial_j u + b(x,u,\partial u)$;where $x = (x_1, x_2, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $n \geq 2$ and $u \in C^2(\Omega)$. The coefficients $(x,z,p),(i,j=1,\dots,n)$, $b(x,z,p)$ are assumed to be real valued and defined for all values of (x,z,p) in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, further $a_{aj} = a_{ij}$, Ω is bounded.

Theorem 3.3.1 Let $u, v\, C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy; $Q(u) \geq Q(v) in\, \Omega$, $u \leq v$ on $\partial\Omega$, where;

i- The operator Q is elliptic;

ii- The coefficients $a_{ij}(x,z,p)$are independent of z.

iii- The coefficients $b(x,y,p)$ is non-increasing in z for each $(x,p) \in \Omega \subset \mathbb{R}^n$.

iv- The coefficients a_{ij}, b are continuously differentiable with respect to the variable p in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Then it follows that u, v in Ω.

In [16] there are formulated conditions such that the Dirichlet problem $Q(u) = 0$ in $\Omega.$; $u = \varphi$ on Ω. Has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ for any $\varphi \in C(\partial\Omega)$ see theorem 15.18 of [9].

Theorem3.3.2: Assume that the above conditions (i)–(iv) of Theorem (3.3.1) are fulfilled with hypothesis of Theorem (15.18) of [16] Then there exists a unique solution of problem (15), (16).

Proof 3: Denote by $G(\varphi)$ the solutionu of the Dirichlet problem $Q(u) = 0$ in Ω; $u = \varphi$ on Ω. Define operator B by; $B(\varphi)(x) \coloneqq h\left(x, G(\varphi)\big(\Phi(x)\big)\right)$, Then $B\colon C(\partial\Omega) \to C(\partial\Omega)$ is nonlinear mapping, where $C(\partial\Omega)$ is a complete metric space with the metric $p(\varphi_1, \varphi_2) = sup|\varphi_1 - \varphi_2|$. It is easy to prove that if $\varphi \in C\ (\partial\Omega)$ is a fixed point of B $i.e.$ $B(\varphi) = \varphi$, then $u \coloneqq G(\varphi)$ is a solution of (15),(16), and conversely, if u is a solution of (15) (16), then $\varphi \coloneqq u|_{\partial\Omega}$ is a fixed point of B. Therefore to prove the existence of (3.3.1), (3.3.2) it is sufficient to show that B has a fixed point. This will be a consequence of Banach's fixed point theorem (see [4]).

Now we show that $B\colon C(\partial\Omega) \to C(\partial\Omega)$ is a contraction on $C(\partial\Omega)$ for any $\varphi_1, \varphi_2 \in C(\partial\Omega)$,

$$\rho\big(B(\varphi_1), B(\varphi_2)\big) = sup|B(\varphi_1) - B(\varphi_2)| \leq q\,.\rho(\varphi_1, \varphi_2) \qquad (17),$$

Where q is a nonnegative number < 1 . By We have $[B(\varphi_1)](x) - [B(\varphi_2)](x) = h\big[x, G(\varphi_1)\big(\Phi(x)\big)\big] - h\big[x, G(\varphi_2)\big(\Phi(x)\big)\big]$. Further, by using Lagrange's mean value theorem and the notations. $b_j \coloneqq G(\varphi_j)\big(\Phi(x)\big)$, $(j = 1,2)$ We find that:

$$[B(\varphi_1)](x) - [B(\varphi_2)](x) = \partial_2 h(x, b_2 + \tau b_2)(b_1 - b_2)$$

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$$= \partial_2 h(x, b_2 + \tau b_2) G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))$$

Consequently,

$$|[B(\varphi_1)](x) - [B(\varphi_2)](x)| \leq sup|\partial_2 h||G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))|$$

We shall prove that: $|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))| \leq \rho(\varphi_1, \varphi_2)$; Where $q := (sup|\partial_2 h| < 1$ is satisfied , then we shall have $\rho(B(\varphi_1) - B(\varphi_2)) \leq q\rho(\varphi_1, \varphi_2)$; This means that B is a contraction in $C(\partial\Omega)$. By using the comparison principle A we want to prove that for all $y \in \Omega$;

$$|G(\varphi_1)(y) - G(\varphi_2)(y)| sup_{\partial\Omega}|\varphi_1 - \varphi_2| \qquad (18)$$

For $u_1 := G(\varphi_1)$, $u_2 := G(\varphi_2)$ then we have: $Q(u_1) = Q(u_2) = 0 \; in \; \Omega$, $u_1|_{\partial\Omega} = \varphi_1$, $u_2|_{\partial\Omega} = \varphi_1$, By using notation $\varepsilon := \sup_{\partial\Omega}|\varphi_1 - \varphi_2|$we may write $\varphi_1 - \varepsilon \leq \varphi_2 \leq \varphi_1 + \varepsilon$.

Consider the function $u := u_2$, $v := u_1 + \varepsilon$ then: $Q(u) = Q(u_2) = 0$ and $Q(v)$ $Q(u_1 + \varepsilon) = \sum_{i,j=1}^{n} a_{ij}(x, \partial(u_1 + \varepsilon))\partial_i(u_1 + \varepsilon)\partial_j(u_1 + \varepsilon) + b(x, u_1 + \varepsilon, \partial(u_1 + \varepsilon)) \leq \sum_{i,j=0}^{n} a_{ij}(x, \partial u_1)(\partial_i u_1)(\partial_j u_1) + b(x, u_1, \partial u_1) = Q(u_1) = 0$,

So we have: $Q(v) = Q(u_1 + \varepsilon) \leq 0 = Q(u_2) = Q(u)$ in Ω, i.e $Q(v) \leq Q(u) \; in \; \Omega$ and $v = u_1 + \varepsilon = \varphi_1 + \varepsilon \geq \varphi_2 = u_2 = u \; on \; \partial\Omega$.

It means that all conditions of comparison principle are fulfilled, this implies that $u \leq v$ inΩ , i.e for all $y \in \Omega$. $u_2(y) \leq u_1(y) + \varepsilon$; Similarly can proved that for all $y \in \Omega$; $u_1(y) - \varepsilon \leq u_2(y)$, thus: $|u_2(y) - u_1(y)| \leq \varepsilon = sup_{\partial\Omega}|\varphi_1 - \varphi_2|$; Thus, we have shown (18) complete the proof of the theorem.

Theorem 3.3.3 Assume that Q satisfies the conditions of theorem (15.18) of [6] and $\Phi: \partial\Omega \to \partial\Omega$ are continuous mapping, h is a continuous function with property $sup|\partial_2 h| < 1$, then there exists a solution of the problem (15)(16) .

Proof : The proof of the Theorem 3.3.3. is similar to the proof of Theorem 3.3.2. except of the proof of equation.(17) , If $\Phi: \partial\Omega \to \partial\Omega, \Psi: \partial\Omega \to \partial\Omega$, then $x \in \partial\Omega$ implies that $\Phi(x) \in \partial\Omega$, and so $G(\varphi_1)(\Phi(x)) = \varphi_1(\Phi(x)), G(\varphi_2)(\Phi(x)) = \varphi_2(\Phi(x))$, And thus: $|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))| = |\varphi_1(\Phi(x)) - \varphi_2(\Phi(x))| \leq \rho(\varphi_1, \varphi_2)$; So the proof can be continued in the way.

Remark 3.3.4 If the condition: $(sup|\partial_2 h| < 1$; is not fulfilled then the nonlocal boundary value problem may have no solution or it may have several solutions.

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4.1 Nonlinear Elliptic Equation With Nonlinear Integral Condition on The Boundary

Consider the following problem:

$$\begin{cases} Q(u) := \sum_{i,j=0}^{n} a_{ij}(x,u,\partial u)\partial_i\partial_j u \; + \; b(x,u,\partial u) = 0 \; in \; \Omega & (19) \\ u(x) = h_1\left(x, u\big(\Phi(x)\big)\right) + \int_{\partial\Omega} h_2\left(x,t,u\big(\Psi(t)\big)\right) \; on \; \partial\Omega & (20) \end{cases}$$

where $\Phi, \Psi: \partial\Omega \to \overline{\Omega}$ are continuous mapping and $h_1 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, $h_2 : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that $|\partial_2 h_1|$, $|\partial_3 h_2|$ exist with the property: $[(sup|\partial_2 h_1| + \lambda(\partial\Omega)\, sup|\partial_3 h_2|) < 1]$, $\lambda(\partial\Omega)$; is the measure of surface $\partial\Omega$.

We shall prove existence and uniqueness of the solution of problem (19),(20) by using argument of [7]. The main result of this paragraph is following.

Theorem4.1.1 Assume that the above conditions (i)–(iv) of Theorem 3.3.1 are fulfilled with hypothesis of Theorem (15.18) of [16] Then there exists a unique solution of (19), (20).

Proof: Denote by $G(\varphi)$ the solution of the Dirichlet proble: $Q(u) = 0$ in Ω.; $u = \varphi$ on Ω. further define operator B by; $B(\varphi)(x) := h_1\left(x, G(\varphi)\big(\Phi(x)\big)\right) + \int_{\partial\Omega} h_2(x,t,G(\varphi)\big(\Psi(t)\big)\; d\sigma_t$,Then $B: C(\partial\Omega) \to C(\partial\Omega)$ is a nonlinear mapping, where $C(\partial\Omega)$ is a complete metric space with the metric $p(\varphi_1,\varphi_2) := sup|\varphi_1 - \varphi_2|$. It is easy to prove that if $\varphi \in C\,(\partial\Omega)$ is a fixed point of B $i.e.$ $B(\varphi) = \varphi$, then $u := G(\varphi)$ is a solution of (3) (4), and conversely, if u is a solution of (3), (4), then $\varphi := u|_{\partial\Omega}$ is a fixed point of B. Therefore to prove the existence of (3), (4) it is sufficient to show that B has a fixed point. This will be a consequence of Banach's fixed point theorem.

Now we show that $B: C(\partial\Omega) \to C(\partial\Omega)$ is a contraction on $C(\partial\Omega)$ for any $\varphi_1, \varphi_2 \in C(\partial\Omega)$, $p\big(B(\varphi_1), B(\varphi_2)\big) = sup|B(\varphi_1) - B(\varphi_2)| \le q.p(\varphi_1,\varphi_2)$, (21)

Where: $q := (sup|\partial_2 h_1| + \lambda(\partial\Omega).sup|\partial_3 h_2| < 1$; We have

$$[B(\varphi_1)](x) - [B(\varphi_2)](x) = \left\{h_1\left[x, G(\varphi_1)\big(\Phi(x)\big)\right] + \int_{\partial\Omega} h_2\left[x,t,G(\varphi_1)\big(\Psi(t)\big)\right]d\sigma_t\right\} - \left\{h_1\left[x, G(\varphi_2)\big(\Phi(x)\big)\right] - \int_{\partial\Omega} h_2\left[x,t,G(\varphi_2)\big(\Psi(t)\big)\right]d\sigma_t\right\}.$$

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Further, by using lagrange's mean value theorem and the notations.

$a_j := G(\varphi_j)(\Phi(x))\,, b_j := G(\varphi_j)(\Psi(t))\,, (j = 1,2)$; We find that:

$[B(\varphi_1)](x) - [B(\varphi_2)](x) = \ \partial_2 h_1(x, a_2 + c[a_1 - a_2])(a_1 - a_2)$

$+ \int_{\partial\Omega} \partial_3 h_2(x, t\,, b_2 + \tilde{c}\,[b_1 - b_2](b_1 - b_2) d\sigma_t$, Consequently,

$|[B(\varphi_1)](x) - [B(\varphi_2)](x)| \leq sup|\partial_1 h_1| \big|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))\big| + sup|\partial_1 h_1|. \int_{\partial\Omega} \big|G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))\big| d\sigma_t,$

We shall prove that:

$$\begin{aligned} \big|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))\big| \leq \rho(\varphi_1, \varphi_2) \\ \big|G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))\big| \leq \rho(\varphi_1, \varphi_2) \end{aligned} \qquad (22)$$

From these inequalities it follows: $\rho\big(B(\varphi_1), B(\varphi_2)\big) \leq q.\rho\ (\varphi_1, \varphi_2)$, where: $q := (sup|\partial_2 h_1| + \lambda(\partial\Omega).\, sup|\partial_3 h_2|$. This means that B is a contraction in $C(\partial\Omega)$. By using conditions of theorem A we want to prove that for all $y := \Phi(x) \in \overline{\Omega}$; $|G(\varphi_1)(y) - G(\varphi_2)(y)| sup|\varphi_1 - \varphi_2|$.

Let $u_1 := G(\varphi_1)\ \ , u_2 := G(\varphi_2)$, then we have $Q(u_1) = Q(u_2) = 0\ in\ \Omega\ , u_1 = \varphi_1\ \ , u_2 = \varphi_1\ on\ \partial\Omega$, We shall show that this implies: $|u_1(y) - u_2(y)| \leq sup_{\partial\Omega}|\varphi_1 - \varphi_2|\ for\ all\ y \in \Omega$. By using notation $\varepsilon := sup_{\partial\Omega}|\varphi_1 - \varphi_2|$we may write $\varphi_1 - \varepsilon \leq \varphi_2 \leq \varphi_1 + \varepsilon$.Consider the function $u := u_2\ \ ,\ v := u_1 + \varepsilon$. since;

$Q(u_1 + \varepsilon) = \sum_{i,j=1}^{n} a_{ij}\big(x, \partial(u_1 + \varepsilon)\big)\partial_i(u_1 + \varepsilon)\partial_j(u_1 + \varepsilon) + b\big(x, u_1 + \varepsilon, \partial(u_1 + \varepsilon)\big) \leq \sum_{i,j=0}^{n} a_{ij}(x, \partial u_1)(\partial_i u_1)\,(\partial_j u_1) + b(x, u_1, \partial u_1) = Q(u_1) = 0$, Thus: $Q(v) = Q(u_1 + \varepsilon) \leq 0 = Q(u_2) = Q(u)\ \ in\ \ \Omega$, Further: $v = u_1 + \varepsilon = \varphi_1 + \varepsilon \geq \varphi_2 = u_2 = u\ \ on\, \partial\,\Omega$. It means that all conditions of Theorem A are fulfilled, thus $u \leq v$ inΩ, i.e for all $y \in \Omega$. $u_2(y) \leq u_1(y) + \varepsilon$, Similarly can proved that for all $y \in \Omega$: $u_1(y) - \varepsilon \leq u_2(y)$; And so we have : $|u_1(y) - u_2(y)| \leq \varepsilon$,Thus we have shown that:

$$\begin{aligned} \big|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))\big| \leq sup|\varphi_1 - \varphi_2|\ = \rho(\varphi_1, \varphi_2) \\ \big|G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))\big| \leq sup|\varphi_1 - \varphi_2|\ = \rho(\varphi_1, \varphi_2) \end{aligned}$$

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Hence we obtain which completes the proof of (Theorem 4.1.1). Since the operator b has exactly one fixed point thus the solution of (19), (20) is unique.

Theorem 4.1.2 Assume that Q satisfies the conditions of theorem (15.18) of [16] and $\Phi, \Psi: \partial\Omega \rightarrow \partial\Omega$ are continuous mapping, h_1, h_2 satisfy the same conditions as in (Theorem 4.1.1), then there exists a unique solution of (19), (20).

The proof of (Theorem 4.1.2) is similar to the proof of (Theorem 4.1.1) except of the proof of eq. (22), since; $\Phi: \partial\Omega \rightarrow \partial\Omega, \Psi: \partial\Omega \rightarrow \partial\Omega,$ thus for $x \in \partial\Omega$ we have: $G(\varphi_1)\big(\Phi(x)\big) = \varphi_1\big(\Phi(x)\big), G(\varphi_2)\big(\Phi(x)\big) = \varphi_2\big(\Phi(x)\big),$ And $G(\varphi_1)\big(\Psi(t)\big) = \varphi_1\big(\Psi(t)\big), G(\varphi_2)\big(\Psi(t)\big) = \varphi_2\big(\Psi(t)\big)$ And so (22) is trivially valid.

Remark 4.1.3 If the condition: $(sup|\partial_2 h_1| + \lambda(\partial\Omega).sup|\partial_3 h_2| < 1$. Is not fulfilled then the nonlocal boundary value problem may have no solution or it may have several solution see [7]. We shall consider examples when the condition $sup|\partial_2 h| < 1$is not fulfilled ,we shall that , then the nonlocal boundary value problem may have no solution or the solution may be not unique.

Example 4.1.4

Case$(n = 1)$:

$$u'' = 0 \ , in\ (0\ ,1), \tag{23}$$

$$u(0) = \ a_0\, u(\alpha) + b_0, \tag{24}$$

$$u(1) = a_1\, u(\beta) + \ b_1, \tag{25}$$

where a_j, b_j constants $(j = 0)$ and $\alpha, \beta \in [0,1]$, we know that all solutions of (23) can be given by; $u(x) = cx + d$, This function satisfies condition (24), (25) if and only if;

$$u(0) = d = \ a_0 u(\alpha) + b_0 = a_0(c\alpha + d) + \ b_0$$

$$u(1) = c + d = \ a_1 u(\beta) + \ b_1 = a_1(c + d) + b_1,$$

So, u satisfies condition (24), (25) if and only if $c\ , d$ satisfy the following system of equations; $a_0(c\alpha + d) + \ b_0 = d, \ \ a_1(c + \beta d) + b_1 = c + d$, This is $(a_0\alpha)c + (a_0 - 1)d = \ -b_0;\ (a_1\beta)c +$

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$(a_1-1)d = -b_0$ And the determinate of this system is; $\begin{vmatrix} a_0\alpha & a_0-1 \\ a_1\beta & a_1-1 \end{vmatrix} = a_0\alpha(a_1-1)-(a_0-1)(a_1\beta-1)$, We shall that if the condition $|a_j|<1$ $(j=0\,,1)$, Is not fulfilled, then $\alpha,\beta\in[0,1]$ can be chose such that the determinate of the system will be 0, in this case the system may have no solution or it may have an infinite number of solutions.

Special cases

If $a_j=1,(j=0,1)$ then the determinate, $h_0(x,z)=z+b_0$, $\partial_2 h_0(z)=1$, $h_1(x,z)=z+b_1$, $\partial_2 h_1(z)=1$, For any $\alpha,\beta\in[0,1]$. If $a_j>1$ $(j=0,1)$, In this case also $\alpha,\beta\in[0,1]$ can be chosen such that the above determinate is 0. $\beta=1$, then the determinate is 0 e.g if $a_0\alpha(a_1-1)=(a_0-1)(a_1-1)$, i.e $a_0\alpha=a_0-1$ And so we obtain $\alpha := \frac{a_0-1}{a_0}\in(0,1)$

Example 4.1.5

(case $n=2$):

$$\Delta u + cu = 0 \quad in \quad B_{1,2}, c\le 0, \qquad (26)$$

where $B_{1,2} := \{x\in R^n : 1<|x|<2\}$ and the boundary condition on;

$$S_j := \{x\in R^n : |x| = j\}, \qquad j=1,2$$
$$u(x) = \beta_1 u(\gamma_1 x)+\delta_1 \quad , x\in S_1, \qquad (27)$$
$$u(x) = \beta_2 u(\gamma_2 x)+\delta_2 \quad , x\in S_2, \qquad (28)$$

where; $1\le\gamma_1\le 2$, $\frac{1}{2}\le\gamma_2\le 1$, Introduce polar coordinates r,θ $in\ \mathbb{R}^n$ such that; $x_1 = r\cos\theta, x_2=\sin\theta$, then we have a new unknown function defined by: $u(r,\theta) := u(x_1,x_2) = u\,(r\cos\theta, r\sin\theta)$, Thus; $\Delta u\,(x_1,x_2) = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial^2 u}{\partial\theta^2}\right)\right](r,\theta)$, Define: $V(r) := \int_0^{2\pi} u(r,\theta)\,d\theta$, Assume that is a solution of (26) we have, $\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial^2 u}{\partial\theta^2}\right)\right]+cu=0$, Integrate both terms with respect to θ on (0,2π), thus we obtain $\int_0^{2\pi}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial^2 u}{\partial\theta^2}\right)\right]d\theta + c\int_0^{2\theta} u\,d\theta = 0$, This is equivalent to $\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\int_0^{2\pi}u(r,\theta)\,d\theta\right]+\frac{1}{r^2}\int_0^{2\pi}\frac{\partial^2 u}{\partial\theta^2}d\theta +$

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$c\int_0^{2\pi} u\, d\theta = 0$. Since; $\int_0^{2\pi} \frac{\partial^2 u}{\partial \theta^2} d\theta = \left[\frac{\partial u}{\partial \theta}\right]_0^{2\pi} = 0$, Thus, we obtain that the following ordinary differential equation for(V).

$$\frac{1}{r}(rV)' + cV = 0, \quad \text{or} \quad V'' + \frac{1}{r}V' + cV = 0 \qquad (29)$$

Thus the solution of equation (29) can be written in the form, $V = d_1V_1 + d_2V_2$; d_1, d_2 are constants, And the boundary condition for (V) is;

$$u(1,\theta) = \beta_1 u(\gamma_1,\theta) + \delta_1, \qquad u(2,\theta) = \beta_2 u(\gamma_2,\theta) + \delta_2,$$

Integrating these equations with respect to θ, we obtain;

$\int_0^{2\pi} u\,(1,\theta)\, d\theta = \beta_1 \int_0^{2\pi} u(\gamma_1,\theta)\, d\theta + 2\pi\delta_1$, $\int_0^{2\pi} u\,(2,\theta)\, d\theta = \beta_2 \int_0^{2\pi} u(\gamma_2,\theta)\, d\theta + 2\pi\delta_2$, This is:

$$V(1) = \beta_1 V(\gamma_1) + 2\pi\delta_1, \qquad (30)$$

$$V(2) = \beta_2 V(\gamma_2) + 2\pi\delta_2, \qquad (31)$$

Thus, we have shown that if u is a solution of (30),(31), then V satisfies (29) ,(30), (31). Now we show that constants β_j, γ_j can be chosen such that; $1 \le \gamma_1 \le 2$, $\frac{1}{2} \le \gamma_2 \le 1$.

And problem (29),(30) has no solution. From this it follows that also (26),(27) has no solution. V is a solution of (30),(31) if and only if d_1, d_2 satisfy the following system of equations:

$$d_1V_1(1) + d_2V_2(1) = \beta_1 d_1V_1(\gamma_1) + \beta_1 d_2V_2(\gamma_1) + 2\pi\delta_1,$$
$$d_1V_1(2) + d_2V_2(2) = \beta_2 d_1V_1(\gamma_2) + \beta_2 d_2V_2(\gamma_2) + 2\pi\delta_1,$$
$$d_1[V_1(1) - \beta_1V_1(\gamma_1)] + d_2[V_2(1) + \beta_1V_2(\gamma_1)] = 2\pi\delta_1,$$
$$d_1[V_1(2) - \beta_2V_1(\gamma_2)] + d_2[V_2(2) + \beta_2V_2(\gamma_2)] = 2\pi\delta_2,$$

And the determinate of this system is:

$$\begin{vmatrix} V_1(1) - \beta_1V_1(\gamma_1) & V_2(1) + \beta_1V_2(\gamma_1) \\ V_1(2) - \beta_2V_1(\gamma_2) & V_2(2) + \beta_2V_2(\gamma_2) \end{vmatrix}, \qquad (32)$$

Special cases: If $\beta_1 = 1$, $\gamma_1 = 1$ then the determinate (32) is 0, and problem (30),(31),(32) may have no solution. In the last case function $u(x)$ are defined by:

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$u(x) = u\ (r, \theta) = \frac{1}{2\pi} V(r)$, will be solution of (26), and thus for our problem (26) we get several solutions. Let γ_1, γ_2 and β_1 be chosen such that;

$$\beta_1[V_2(\gamma_2)V_2(\gamma_1) - V_1(\gamma_2)V_2(\gamma_1)] + V_1(\gamma_2)V_2(1) - V_2(\gamma_2)V_1(1) \neq 0$$

Then we can be chosen such that the determinate (32) is equal to 0, and this problem (26) will have no solution or it will have several solutions.

Results

Nonlocal Boundary Conditions: The paper explores the Laplace equation with nonlocal boundary conditions, establishing a connection between the trace of an unknown function on a manifold and its values at different points, which is crucial for understanding the behavior of solutions in nonlocal settings.

- **Sobolev Spaces**: It discusses several basic definitions and properties of Sobolev spaces, which are essential for analyzing the solutions of partial differential equations. The paper emphasizes the importance of these spaces in the context of nonlocal boundary value problems**.**
- Existence and Uniqueness: The results include theorems that address the existence and uniqueness of solutions for the nonlocal elliptic boundary value problems, providing a foundation for further research in this area**.**
- Trace Theorems: The research highlights the significance of trace theorems, which relate to the behavior of functions at the boundary, thereby facilitating the formulation of nonlocal boundary conditions.
- Imbedding Theorems: The paper presents results on the imbedding of function spaces, which are vital for ensuring that solutions to the boundary value problems are well-defined and can be effectively analyzed within the framework of Sobolev spaces.

Conclusion

- This research stretched essential contributions to the adissection of nonlocal boundary value problems for elliptic equations via four point findings:

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- 1. Expounded the stringent role of nonlocal boundary conditions in modeling complex physical systems.
- 2. Founded Sobolev spaces as the superior functional setting of analytic solution.
- 3. Confirm strict existence and uniqueness theorems under practical terms.
- 4. Developed novel depiction theorems joining boundary conduct with solution estates.

Future research directions should focus on

- - Expanding outlooks to nonlinear and fractional operators.
- - Layout particular numeration methods.
- - Scouting implementations in preceding substances and biological programs

These results exemplify fundamental proceeds in conception nonlocal phenomena and supply worthy machines for researchers in applied mathematics and physics. The evolved scope shows new chances for analyzing complex systems through twofold precisions.

Research Recommendations

- For Theoretical Expansion, We counsel expanding these outlooks to nonlinear P-Laplacian operators, and it's important to study solution estates in fractional sobolev spaces.
- For Practical Applications, These outlooks can be stratified to modeling nonlocal estates of intelligent substances.

Future Research Directions

- Examining unsettled (local/nonlocal) boundary value problems is proposed.
- Conjunction this outlook with profound learning techniques advantage scouting.

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